Design of Robust Constrained Model-Predictive Controllers with Volterra Series

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Sufficient conditions are developed in this work for robust stability and performance of nonlinear model-predictive control systems that use an end-condition and second-order Volterra model with parametric uncertainty in the time domain. The robust stability conditions involve the lengths of the prediction and control horizons, as well as the coefficients of the control move suppression terms in the on-line objective function. These conditions may be used for both analysis and synthesis purposes. A case study of a chemical reactor is presented to elucidate these issues.

Introduction

The fact that most chemical processes are nonlinear is well documented in the literature (Shinskey, 1967; Foss, 1973; Buckley, 1981; Garcia and Prett, 1986; Morari, 1986; NRC Committee Report, 1988; Fleming, 1988; Prett and Garcia, 1988; Edgar, 1989; Longwell, 1991; Bequette, 1991; Kane, 1993). However, linear controllers are very commonly used with nonlinear processes because of their simple implementation and analysis. Linear model predictive controllers (MPC) such as dynamic matrix control (DMC) (Cutler and Ramaker, 1980) can perform well with several nonlinear processes, since feedback control tends to decrease the effects of nonlinearity. In fact, the reduction of nonlinearity was one of the reasons why feedback control was first employed (Black, 1977; Nikolaou, 1993). Yet, control of complex nonlinear process operations may necessitate the use of MPC based on a nonlinear model. Progress in nonlinear MPC not only would result in better process control, but could also allow many processes to be designed and operated closer to more complex (but optimal) operating regimes (Seider et al., 1990, 1991). Although several studies have addressed computational issues of nonlinear MPC with constraints, theoretical investigations have appeared only recently (Rawlings et al., 1994). In particular, there have been, to our knowledge, only a few published results on robust stability and performance of constrained MPC with nonlinear models (de Oliveira and Morari, 1994; Michalska and Mayne, 1993). This work provides some new results in this area. More specifically, our main objective in this work is to address the following question: *How can model-predictive controllers be rigorously designed for nonlinear processes, so that robust stability and performance can be ensured in the presence of process input/output constraints and modeling uncertainty?*

A corollary question is: *How much modeling accuracy is required for robust stability and performance of constrained nonlinear MPC?*

Clearly, a methodology that encompasses all dynamic systems that are not linear (hence, by default, nonlinear) would be a desirable but also formidable, if not unmanageable, task. In this work we focus our attention on dynamic systems that are modeled by Volterra series (Schetzen, 1980; Boyd and Chua, 1985). Some advantages of such models are:

- Coefficients appear linearly.
- Modeling uncertainty can be quantified as confidence intervals for the coefficients of the model (Doyle et al., 1992; Pearson et al., 1992).

Recent results have put the robust constrained MPC design for linear processes in a rigorous framework (Meadows and Rawlings, 1993; Genceli and Nikolaou, 1993; Zheng and Morari, 1993).

For constrained MPC with nonlinear models, most efforts have concentrated on numerical aspects of the on-line nonlinear optimization that MPC performs (Biegler and Rawlings, 1991; Gattu and Zafiriou, 1992; Sistu et al., 1993). In a recent theoretical development, Meadows and Rawlings (1993) use a state-space approach to derive stability conditions for constrained nonlinear MPC systems without modeling uncertainty, and demonstrate that unexpected difficulties may lurk behind seemingly well-behaved nonlinear systems. A survey of the state of the art on nonlinear MPC is given by Rawlings et al. (1994).

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The approach that we take in this article relies on an extension of a technique that proved successful in our previous work with constrained MPC of open-loop stable linear processes. Some features of our approach that proved useful in the pursuit of robustness conditions for nonlinear constrained MPC follow:

- Minimization of an on-line objective function based on the l_1-norm (absolute values of errors and input changes). Numerical solution of that on-line minimization can be efficiently obtained through sequential linear programming (cutting plane method) (Rao, 1984).
- Use of Volterra models. Process modeling uncertainty is quantified as uncertainty in the values of the Volterra model coefficients, which are obtained from identification experiments. This allows robust stability and performance design conditions to be developed in a rigorous manner.
- Softening of all hard constraints on process outputs, through the introduction of slack variables (Zafiriou and Marchal, 1991). This approach is used to prevent infeasibilities for the on-line minimization problem.
- Use of an end-condition, that is, the value of the process input at the end of the finite horizon must be equal to the steady-state input value corresponding to the set-point and disturbance steady-state estimates. The end-condition is useful for showing closed-loop stability with zero offset, for open-loop stable processes.

Controller tuning parameters that appear in the robust stability and performance conditions include:

- The lengths m and p of the control and prediction horizons, respectively;
- Values of the weights r_i in the control move-suppression terms of the on-line objective function.

The rest of this article is structured as follows: The nonlinear dynamic matrix control problem with end-condition (ENLDMC) is formulated first. Then our main results on robust stability and performance of ENLDMC are presented. A case study is then included to illustrate the proposed design methodology based on the main results. Finally, conclusions are drawn and directions for further research are given.

Nonlinear Dynamic Matrix Control with End-Condition

In a typical formulation, linear MPC performs the following on-line minimization:

$$\min_{u(k),\ldots,u(k+m),\epsilon(k+1),\ldots,\epsilon(k+p)} \sum_{j=1}^{p} |\bar{y}(k+j)-y^{sp}|^{l} + w\epsilon(k+j)^{l} + \sum_{i=0}^{m} r_i|\Delta u(k+i)|^{l}, \quad l=1,2 \tag{1}$$

subject to

$$y_{\min}-\epsilon(k+j)\le \bar{y}(k+j)\le y_{\max}+\epsilon(k+j),$$

$$\epsilon(k+j)\ge 0 \qquad j=1,\ldots,p \tag{2}$$

$$u_{\min}\le u(k+i)\le u_{\max}, \qquad i=0,\ldots,m \tag{3}$$

$$-\Delta u_{\max}\le \Delta u(k+i)\le \Delta u_{\max} \qquad i=0,\ldots,m \tag{4}$$

$$\Delta u(k+m+i)=0 \qquad i\ge 1, \tag{5}$$

where a linear model is used at time k to predict future outputs $\bar{y}(k+j)$. While the preceding relations remain unchanged in nonlinear MPC, a nonlinear model is used to predict $\bar{y}(k+j)$.

In this study we selected $l=1$ (l_1-norm MPC) and we use a second-order Volterra model to represent a nonlinear process. The model is obtainable from input/output correlation data (Doyle et al., 1992). Hard confidence intervals on the model's coefficients are assumed to be known. Such confidence intervals may be obtained either statistically, or by applying deterministic modeling ideas (Chen et al., 1992).

The formulation of the on-line optimization problem for nonlinear MPC with second-order Volterra series, that we propose is presented next. Let

$$y(k)=d(k)+\sum_{j=1}^{N} h_j u(k-j) + \sum_{j_1=1}^{K}\sum_{j_2=1}^{K} \boldsymbol{h}_{j_1,j_2} u(k-j_1)u(k-j_2) \tag{6}$$

describe the real plant behavior, where only the estimates g_j and $\boldsymbol{g}_{j_1,j_2}$ of h_j and $\boldsymbol{h}_{j_1,j_2}$, respectively, are known, that is,

$$h_j=g_j+e_j, \qquad 1\le j\le N \tag{7}$$

$$|e_j|\le E_j, \qquad 1\le j\le N \tag{8}$$

$$\boldsymbol{h}_{j_1,j_2}=\boldsymbol{g}_{j_1,j_2}+\boldsymbol{e}_{j_1,j_2}, \qquad 1\le j_1,j_2\le K \tag{9}$$

$$|\boldsymbol{e}_{j_1,j_2}|\le \boldsymbol{E}_{j_1,j_2}, \qquad 1\le j_1,j_2\le K, \tag{10}$$

where e_j and $\boldsymbol{e}_{j_1,j_2}$ are bounded additive errors.

Prediction through a second-order Volterra model has the form

$$\bar{y}(k+i)=\bar{d}(k+i)+\sum_{j=1}^{N} g_j u(k+i-j) + \sum_{j_1=1}^{K}\sum_{j_2=1}^{K} \boldsymbol{g}_{j_1,j_2} u(k+i-j_1)u(k-i-j_2), \tag{11}$$

with zero-order disturbance model as

$$\bar{d}(k+i)=\bar{d}(k)=y(k)-\sum_{j=1}^{N} g_j u(k-j) - \sum_{j_1=1}^{K}\sum_{j_2=1}^{K} \boldsymbol{g}_{j_1,j_2} u(k-j_1)u(k-j_2)$$

$$=d(k)+\sum_{j=1}^{N} e_j u(k-j) + \sum_{j_1=1}^{K}\sum_{j_2=1}^{K} \boldsymbol{e}_{j_1,j_2} u(k-j_1)u(k-j_2) \tag{12}$$

where $y(k)$ is the current measurement of the process output.

Remark. The zero-order disturbance prediction is a convenient assumption, in the traditional DMC framework, used to include the feedback effect of the measurement $y(k)$. The derivations regarding stability in the sequel will hold whether or not this assumption is true.

Finally we use the end-condition

$$\bar{y}(k+\infty) = y^{sp} = \bar{d}(k) + Gu(k+m) + \mathbf{G}u(k+m)^2, \quad (13a)$$

implying

$$u(k+m) = \frac{-G+\sqrt{\Delta}}{2\mathbf{G}}, \quad \text{if } \mathbf{G} > 0 \left[\text{reject, } u(k+m) = \frac{-G+\sqrt{\Delta}}{2\mathbf{G}}\right] \quad (13b)$$

or

$$u(k+m) = \frac{-G-\sqrt{\Delta}}{2\mathbf{G}}, \quad \text{if } \mathbf{G} < 0 \left[\text{reject, } u(k+m) = \frac{-G+\sqrt{\Delta}}{2\mathbf{G}}\right], \quad (13c)$$

where

$$G = \sum_{i=1}^{N} g_i \quad (14a)$$

$$\mathbf{G} = \sum_{i=1}^{K} \sum_{j=1}^{K} g_{ij} \quad (14b)$$

and

$$\Delta = G^2 - 4\mathbf{G}[\bar{d}(k) - y^{sp}] \quad (14c)$$

$$= G^2 - 4\mathbf{G}\left[d(k) - y^{sp} + \sum_{j=1}^{N} e_j u(k-j) + \sum_{j_1=1}^{K} \sum_{j_2=1}^{K} \mathbf{e}_{j_1,j_2} u(k-j_1)u(k-j_2)\right]. \quad (14d)$$

This completes the formulation of ENLDMC.

Remark. Equation 13a poses a problem that is not present in linear DMC with end-condition (ELDMC) (Genceli and Nikolaou, 1993), namely, which one of the two roots of Eq. 13a to select as the end-condition. As Eqs. 13b and 13c show, the selection of the value of $u(k+p)$ is based on the satisfaction of the following condition:

$$\lim_{\mathbf{G} \to 0} u(k+m) < \infty.$$

Feasibility of ENLDMC On-Line Problem

The ENLDMC on-line problem as defined by Eqs. 1 to 14 has a solution if

$$(1+m)\Delta u_{\max} \geq u_{\max} - u_{\min}, \quad (15)$$

and

$$u_{\max} \geq \frac{-G+\sqrt{\Delta}}{2\mathbf{G}} \geq u_{\min}, \quad \text{if } \mathbf{G} > 0 \quad (16a)$$

$$u_{\max} \geq \frac{-G-\sqrt{\Delta}}{2\mathbf{G}} \geq u_{\min}, \quad \text{if } \mathbf{G} < 0 \quad (16b)$$

for all possible Δ's. In addition, Δ should be nonnegative. From Eq. 14 this condition is guaranteed to be satisfied if

$$|d(k) - y^{sp}| \leq \frac{G^2}{4|\mathbf{G}|} - U\sum_{i=1}^{N} E_i - U^2 \sum_{i=1}^{K}\sum_{j=1}^{K} \mathbf{E}_{ij} \quad (16c)$$

where $U = \max\{|u_{\max}|, |u_{\min}|\}$.

Robust Stability of Closed-Loop ENLDMC

In the sequel, we will derive sufficient conditions for robust closed-loop stability. The proof relies on a Lyapunov type of argument (Genceli and Nikolaou, 1993; Rawlings and Muske, 1993), and capitalizes on the end-condition to ensure zero offset to step disturbances.

Theorem (Robust stability of ENLDMC). For a plant that is described by Eq. 6 and disturbances that satisfy the conditions

$$d_{\min} \leq d(k) \leq d_{\max} \quad (17)$$

$$|\Delta d(k)| \leq \Delta d_{\max}, \qquad k \leq M < \infty \quad (18a)$$

$$|\Delta d(k)| = 0, [d(k) = d_\infty], \; k > M < \infty \quad (18b)$$

the ENLDMC closed-loop system with controller described by the set of relations 1, 2, 3, 4, 5, 11, 12, 13 and 14 is asymptotically stable with zero offset if

(i) Process modeling and disturbance uncertainties satisfy conditions 16 and

$$\left(|G| - 2U|\mathbf{G}| - \sum_{i=1}^{N} E_i - 2U\sum_{i=1}^{K}\sum_{j=1}^{K} \mathbf{E}_{ij}\right)\Delta u_{\max} \geq \Delta d_{\max} > 0. \quad (19)$$

(ii) The control horizon length m and prediction horizon length p satisfy relations 15 and

$$p = m + N. \quad (20)$$

(iii) The move suppression terms $\{r_i\}_{i=0}^{m}$ are selected according to the relations

$$r_0 = \frac{\eta(1+w)(m+1+A) + \sum_{j=-N+1}^{0} \delta_j}{1 - \frac{\eta}{B}} \quad (21)$$

$$r_j = r_{j-1}, \qquad 1 \leq j \leq m, \quad (22)$$

where δ_j is a small nonnegative number,

$$\eta = \sum_{i=1}^{N} E_i + 2U \sum_{i=1}^{K} \sum_{l=1}^{K} \boldsymbol{E}_{i,l} \tag{23}$$

$$B = |G| - 2U|\boldsymbol{G}| \tag{24}$$

$$A = \max_{\hat{u}(k+m+2-N),\hat{u}(k+m+3-N),\ldots,\hat{u}(k+m),\tilde{u}(k+m+1)} \sum_{i=1+m}^{p} |D_i| \tag{25}$$

subject to

$$u_{\max} \geq \hat{u}(k+j) \geq u_{\min}, \qquad m \geq j \geq m+2-N$$

$$\Delta u_{\max} \geq \hat{u}(k+j) - \hat{u}(k+j-1) \geq -\Delta u_{\max}, \qquad m \geq j \geq m+3-N$$

$$u_{\max} \geq \tilde{u}(k+m+1) \geq u_{\min},$$

where

$$D_i = C_i - \frac{GC_i - \sum_{j=i-m+1}^{N} \sum_{l=1}^{i-m} (g_{lj} + \boldsymbol{g}_{jl}) \hat{u}(k+i+1-j)}{G + \boldsymbol{G}[\hat{u}(k+m) + \tilde{u}(k+m+1)]} \tag{26}$$

$$C_i = \frac{\sum_{j=1}^{i-m} \sum_{l=1}^{i-m} \boldsymbol{g}_{jl}}{\boldsymbol{G}} - 1. \tag{27}$$

Proof. Let us denote all past inputs implemented before the sampling time k by $\hat{u}(k-j)$ where $j \geq 1$. Assuming that the feasibility conditions (Eqs. 15 and 16) are satisfied, let

$$\{\hat{\epsilon}(k+1),\ldots,\hat{\epsilon}(k+p),\hat{u}(k),\hat{u}(k+1),\ldots,\hat{u}(k+m)\} \tag{28}$$

be the optimal solution of Eq. 1 with $l_1 = 1$ corresponding to the optimal predicted outputs at time k

$$\{\hat{y}(k+1),\ldots,\hat{y}(k+p)\}.$$

It is clear that $\{\hat{\epsilon}(k+1),\ldots,\hat{\epsilon}(k+p),\ \hat{u}(k),\ \hat{u}(k+1),\ldots,\hat{u}(k+m)\}$ will also be an optimal solution for the problem

$$\min_{\epsilon(k+1),\ldots,\epsilon(k+p),u(k),u(k+1),\ldots,u(k+m)} J(k) \tag{29}$$

subject to the constraints in Eqs. 2 to 5, with

$$J(k) = w\hat{\epsilon}(k) + w \sum_{j=1}^{p} \epsilon(k+j) + |y(k) - y^{sp}| + \sum_{j=1}^{p} |\bar{y}(k+j) - y^{sp}| + \sum_{j=-N+1}^{m} r_i |\Delta u(k+j)| + f(k) \tag{30}$$

where

$$\hat{\epsilon}(k) = \max([y(k) - y_{\max}],[y_{\min} - y(k)],0) \tag{31}$$

and $f(k)$ is an arbitrary function of k.

Hence, at time k, the optimal value of $J(k)$, $\Phi(k)$, is

$$\Phi(k) = w\hat{\epsilon}(k) + w \sum_{j=0}^{p} \hat{\epsilon}(k+j) + |y(k) - y^{sp}| + \sum_{j=1}^{p} |\bar{y}(k+j) - y^{sp}| + \sum_{j=-N+1}^{m} r_j |\Delta \hat{u}(k+j)| + f(k). \tag{32}$$

The main idea of the proof is to develop conditions that ensure that the sequence $\{\Phi(k)\}_{k=0}^{\infty}$ converges. To do that, observe that a suboptimal cost $\Phi^*(k+1)$ at time $k+1$ satisfies the following inequalities:

$$0 \leq \Phi(k+1) \leq \Phi^*(k+1) \tag{33}$$

$$\Phi(k) \geq \Phi(k+1) + (\Phi(k) - \Phi^*(k+1)) \tag{34}$$

where

$$\Phi(k+1) = \min_{\epsilon(k+2),\ldots,\epsilon(k+1+p),u(k+1),u(k+2),\ldots,u(k+1+m)} J(k+1) \tag{35}$$

subject to ENLDMC constraints. If we find conditions that make $\Phi(k) - \Phi^*(k+1)$ nonnegative, then we will have from Eqs. 33 and 34 that $\Phi(k) \geq \Phi(k+1) \geq 0$, which will guarantee the convergence of $\{\Phi(k)\}_{k=0}^{\infty}$. To do that, let us try to create a feasible (but not necessarily optimal) solution to Eq. 35 as

$$\{\tilde{\epsilon}(k+2),\ldots,\tilde{\epsilon}(k+1+p),\tilde{u}(k+1),\tilde{u}(k+2),\ldots,\tilde{u}(k+m), \tilde{u}(k+1+m)\} = \{\hat{\epsilon}(k+2) + |\hat{y}(k+2) - \tilde{y}(k+2)|,\ldots, \hat{\epsilon}(k+p) + |\hat{y}(k+p) - \tilde{y}(k+p)|, 0, \hat{u}(k+1),\hat{u}(k+2),\ldots, \hat{u}(k+m),\tilde{u}(k+1+m)\} \tag{36}$$

where only the last term, $\tilde{u}(k+1+m)$, is to be selected. The corresponding predicted future outputs at time $k+1$ are denoted by

$$\{\tilde{y}(k+2),\ldots,\tilde{y}(k+1+p)\},$$

yielding a suboptimal cost $\Phi^*(k+1)$.

First, let us obtain sufficient conditions for which the set (Eq. 36) is a feasible solution of Eq. 35.

Feasibility of u and Δu. It is clear that all $\hat{u}$'s and $\Delta\hat{u}$'s satisfy the ENLDMC constraints because they satisfied them in the previous sampling time k. Hence $\tilde{u}(k+1+m)$ remains to be selected so that it can satisfy the constraint 4, that is,

$$\Delta u_{\max} \geq \Delta\tilde{u}(k+1+m) \geq -\Delta u_{\max}, \quad (37)$$

the constraint 3, that is

$$u_{\min} \leq \tilde{u}(k+1+m) \leq u_{\max}, \quad (38)$$

and the following end-condition (derived from the combination of Eq. 12 with Eq. 13a):

$$\tilde{y}(\infty) = y^{sp} = d(k+1) + G\tilde{u}(k+1+m) + \boldsymbol{G}\tilde{u}(k+1+m)^2 + \sum_{i=1}^{N} e_i\hat{u}(k+1-i) + \sum_{j_1=1}^{K}\sum_{j_2=1}^{K} \boldsymbol{e}_{j_1,j_2} \times \hat{u}(k+1-j_1)\hat{u}(k+1-j_2). \quad (39)$$

Combining Eqs. 38 and 39, we get, equivalently

$$u_{\max} \geq \frac{-G+\sqrt{\boldsymbol{\Delta}_2}}{2\boldsymbol{G}} \geq u_{\min}, \qquad \text{if } \boldsymbol{G} > 0$$

$$u_{\max} \geq \frac{-G-\sqrt{\boldsymbol{\Delta}_2}}{2\boldsymbol{G}} \geq u_{\min}, \qquad \text{if } \boldsymbol{G} < 0,$$

where

$$\boldsymbol{\Delta}_2 = G^2 - 4\boldsymbol{G}\left[d(k+1) - y^{sp} + \sum_{j=1}^{N} e_j\hat{u}(k+1-j) + \sum_{j_1=1}^{K}\sum_{j_2=1}^{K} \boldsymbol{e}_{j_1,j_2}\hat{u}(k+1-j_1)\hat{u}(k+1-j_2)\right].$$

The preceding condition is satisfied if conditions 16 are true. To satisfy the constraint (Eq. 37), we start by subtracting

$$\hat{y}(\infty) = y^{sp} = d(k) + G\hat{u}(k+m) + \boldsymbol{G}\hat{u}(k+1+m)^2 + \sum_{i=1}^{N} e_i\hat{u}(k-i) + \sum_{j_1=1}^{K}\sum_{j_2=1}^{K} \boldsymbol{e}_{j_1,j_2}\hat{u}(k-j_1)\hat{u}(k-j_2) \quad (40)$$

from Eq. 39 to get

$$\Delta\tilde{u}(k+1+m) = \frac{-\Delta d(k+1) - \sum_{j=1}^{N} e_j\Delta\hat{u}(k+1-j) - \sum_{j_1=1}^{K}\sum_{j_2=1}^{K}\boldsymbol{e}_{j_1,j_2}\Delta[\hat{u}(k+1-j_1)\hat{u}(k+1-j_2)]}{G+\boldsymbol{G}[\tilde{u}(k+m+1)+\hat{u}(k+m)]}. \quad (41)$$

The $\Delta\tilde{u}(k+1+m)$ given earlier must satisfy Eq. 37. This is guaranteed if condition 19 is true.

Feasibility of ε. For the feasibility of $\tilde{\epsilon}(k+j)$'s we have

$$y_{\min} - \hat{\epsilon}(k+1+j) \leq \hat{y}(k+1+j), \quad 1 \leq j \leq p-1 \quad (42)$$

or, equivalently,

$$y_{\min} - \hat{\epsilon}(k+1+j) - |\hat{y}(k+1+j) - \tilde{y}(k+1+j)| \leq \hat{y}(k+1+j) - |\hat{y}(k+1+j) - \tilde{y}(k-1+j)| \leq \tilde{y}(k+1+j), \quad 1 \leq j \leq p-1. \quad (43)$$

Similarly,

$$y_{\max} + \hat{\epsilon}(k+1+j) \geq \hat{y}(k+1+j), \quad 1 \leq j \leq p-1 \quad (44)$$

or, equivalently,

$$y_{\max} + \hat{\epsilon}(k+1+j) + |\hat{y}(k+1+j) - \tilde{y}(k+1+j)| \geq \hat{y}(k+1+j) + |\hat{y}(k+1+j) - \tilde{y}(k+1+j)| \geq \tilde{y}(k+1+j), \quad 1 \leq j \leq p-1 \quad (45)$$

and, finally, $\tilde{\epsilon}(k+1+p) = 0$ is feasible because of the end-condition $\tilde{y}(k+1+p) = y^{sp}$. Therefore Eq. 36 is indeed feasible.

We are now ready to derive closed-loop robust stability conditions, which guarantee that $\{\Phi(k)\}_{k=0}^{\infty}$ is convergent. After some manipulations, inequality (Eq. 34) yields

$$\Phi(k) \geq \Phi(k+1) + |y(k) - y^{sp}| + w\hat{\epsilon}(k) - (1+w)|\hat{y}(k+1) - y(k+1)| - (1+w)\sum_{j=2}^{p}|\hat{y}(k+j) - \tilde{y}(k+j)| - r_m|\Delta\tilde{u}(k+1+m)| + \sum_{j=-N+1}^{m}(r_j - r_{j-1})|\Delta\hat{u}(k+j)| + f(k) - f(k+1) \quad (46)$$

$$r_{-N} = 0.$$

The predicted outputs $\{\hat{y}(k+j),\ j=1,\ldots,p\}$ and $\{\tilde{y}(k+j),\ j=2,\ldots,1+p\}$ can be expressed in terms of the corresponding inputs to yield

$$\hat{y}(k+1) - y(k+1) = -\Delta d(k+1) - \sum_{j=-N+1}^{0} e_{1-j}\Delta\hat{u}(k+j) - \sum_{j_1=-N+1}^{0}\sum_{j_2=-N+1}^{0} \boldsymbol{e}_{1-j_1,1-j_2}\Delta[\hat{u}(k+j_1)\hat{u}(k+j_2)] \quad (47)$$

$$\hat{y}(k+1+i)-\tilde{y}(k+1+i)=-\Delta d(k+1)-\sum_{j=-N+1}^{0} e_{1-j}\Delta\hat{u}(k+j)-\sum_{j_1=-N+1}^{0}\sum_{j_2=-N+1}^{0} e_{1-j_1,1-j_2}\times\Delta[\hat{u}(k+j_1)\hat{u}(k+j_2)],\quad 1\le i\le m \quad (48)$$

$$\hat{y}(k+1+i)-\tilde{y}(k+1+i)=D_i\Delta d(k+1)+\sum_{j=-N+1}^{0} e_{1-j}D_i\Delta\hat{u}(k+j)+\sum_{j_1=-N+1}^{0}\sum_{j_2=-N+1}^{0} e_{1-j_1,1-j_2}D_i\times\Delta[\hat{u}(k+j_1)\hat{u}(k+j_2)],\quad 1+m\le i\le p-1,\quad (49)$$

where D_i and C_i are given by Eqs. 26 and 27. Substituting Eqs. 41, 47, 48 and 49 in Eq. 46 yields, after lengthy manipulations (see supplementary material),

$$\Phi(k)\ge\Phi(k+1)+|y(k)-y^{sp}|+w\hat{\epsilon}(k)+\sum_{j=1}^{m}(r_j-r_{j-1})|\Delta\hat{u}(k+j)|+\sum_{j=-N+1}^{0}\left(r_j-r_{j-1}-\left[(1+w)(m+1+A)+\frac{r_m}{B}\right]\times\left[E_{1-j}+2U\sum_{l=-N+1}^{0}\boldsymbol{E}_{1-j,1-l}\right]|\Delta\hat{u}(k+j)|\right)+\left(f(k)-f(k+1)-\left[(1+w)(m+1+A)+\frac{r_m}{B}\right]\Delta d_{\max}\right) \quad (50)$$

where B and A are given by Eqs. 24 and 25.

As stated before, we would like to make the quantity added to $\Phi(k+1)$ in the righthand side of inequality (Eq. 50) nonnegative, in order to guarantee that the sequence $\{\Phi(k)\}_{k=0}^{\infty}$ is nonincreasing. This is guaranteed if $\{r_j\}_{j=-N+1}^{m}$ and $\{f(k)\}_{k=0}^{\infty}$ are chosen to satisfy the following equalities, for $\delta_j\ge 0$.

$$r_j-r_{j-1}=\delta_j,\quad 1\le j\le m \quad (51)$$

$$r_j-r_{j-1}-\left[(1+w)(m+1+A)+\frac{r_m}{B}\right]\times\left[E_{1-j}+2U\sum_{l=-N+1}^{0}\boldsymbol{E}_{1-j,1-l}\right]=\delta_j,\ -N<j\le 0 \quad (52)$$

$$f(k)-f(k+1)-\left[(1+w)(m+1+A)+\frac{r_m}{B}\right]\Delta d_{\max}=\delta_{-N},\quad 0\le k\le M \quad (53)$$

$$f(k)=0,\quad k>M. \quad (54)$$

The solution of Eqs. 51 to 54 given by Eqs. 21, 22, and the following equations, which are not needed in controller design:

$$r_{j-1}=r_j-\left[(1+w)(m+1+A)+\frac{r_0}{B}\right]\times\left[E_{1-j}+2U\sum_{l=-N+1}^{0}\boldsymbol{E}_{1-j,1-l}\right]-\delta_j,\ -N<j\le 0 \quad (55)$$

$$f(k)=\left[\frac{\delta_{-N}}{\Delta d_{\max}}+(1+w)(m+1+A)+\frac{r_0}{B}\right][M+1-k]\Delta d_{\max},\quad k\ge 0,\quad (f(k)=0,\quad k>M). \quad (56)$$

If the preceding equations are satisfied, then $\{\Phi(k)\}_{k=0}^{\infty}$ is a nonincreasing sequence. It is also bounded below by zero. Therefore $\{\Phi(k)\}_{k=0}^{\infty}$ converges. This implies that taking the limit of Eq. 50, as $k\to\infty$, yields

$$0\ge\lim_{k\to\infty}\left(|y(k)-y^{sp}|+w\hat{\epsilon}(k)\sum_{j=-N+1}^{0}\delta_j|\Delta\hat{u}(k+j)|\right)\ge 0$$

$$\Rightarrow\lim_{k\to\infty}y(k)=y^{sp}.$$

QED.

Remarks

- The significance of Eq. 16c is that the magnitude of disturbances that can be successfully rejected is limited not only by the available control action U, but also process nonlinearity $\boldsymbol{G}$. The less nonlinear the process, the larger the disturbances that can be rejected.
- The significance of Eq. 19 is that for a given allowable range U of inputs u, the nonlinearity cannot exceed a certain magnitude, that is, it must satisfy the inequality

$$|\boldsymbol{G}|\le\frac{\left(|G|-\sum_{i=1}^{N}E_i-2U\sum_{i=1}^{K}\sum_{j=1}^{K}\boldsymbol{E}_{ij}\right)}{2U}.$$

- In contrast to linear robust control theory, which has been developed mostly in the frequency domain, our results rely on parametric uncertainty in the time domain. This allows for convenient characterization of process uncertainty through parameter estimation experiments.
- The robust stability conditions are independent of the numerical method used to perform the on-line optimization dictated by MPC.
- As shown in the Appendix, the on-line MPC optimization problem can be reformulated to have a linear objective with linear and quadratic constraints. The latter problem can be solved through sequential linear programming. The preceding stability analysis is valid even if a local optimum is reached.

Robust Performance of Closed-Loop ENLDMC

Definition. The performance, P, of an ENLDMC closed-loop system is measured by the following quantity:

$$P=\sum_{k=0}^{\infty}[|y(k)-y^{sp}|+w\max(0,[y(k)-y_{\max}],[y_{\min}-y(k)])],$$

Table 1. CSTR Parameters

F	V	C_{Ai}	k	E/R	T_i
m^3/h	m^3	mol/m^3	h^{-1}	K	K
1.133	1.36	8008	7.0e07	8375	373.3
ΔH_R	c_p	ρ	Q_s	T_s	C_{As}
J/mol	J/kg/K	kg/m^3	J/h	K	mol/m^3
−69,775	3,140	800.8	1.055e08	547.6	393.2

where y is the measured process output. The first term in this sum refers to the output error, while the second captures the violation of constraints by the output of the process.

The following corollary is a direct consequence of the main theorem. It is useful in the selection of proper control and prediction horizon lengths m and p, respectively.

Corollary (robust performance of ENLDMC). An ENLDMC controller defined by the set of relations 1 to 5, and satisfying the conditions of the robust stability theorem, achieves a closed-loop performance, P, no worse than the initial on-line calculated optimal objective function value $\Phi(0)$, that is,

$$P \leq \Phi(0) = \min_{\epsilon(1),\ldots,\epsilon(p),u(0),u(1),\ldots,u(m)} J(0)$$

subject to operating constraints, where

$$J(0) = \sum_{j=0}^{p} |\bar{y}(j) - y^{sp}| + w\max(0,[\bar{y}(j) - y_{\max}],[y_{\min} - \bar{y}(j)]) + \sum_{j=0}^{m} r_j|\Delta u(j)| + f(0).$$

Proof. Let us choose $\delta_j = 0$ for $-N \leq j \leq m$. Then, relation 50 becomes

$$\Phi(k) \geq \Phi(k+1) + |y(k) - y^{sp}| + w\max(0,[y(k) - y_{\max}],[y_{\min} - y(k)]).$$

Successive substitution of $\Phi(k)$ in the preceding equation, starting from the initial time $k = 0$, yields

$$\Phi(0) \geq \sum_{k=0}^{\infty} [|y(k) - y^{sp}| + w\max(0,[y(k) - y_{\max}],[y_{\min} - y(k)])] = P.$$

QED.

Table 2. Linear Model with Uncertainty

j	g_j	E_j*	E_j
1	−5.3	0.25	0.1
2	−2.4	0.2	0.1
3	−1.04	0.15	0.1
4	−0.45	0.1	0.05
5	−0.2	0.08	0.04
6	−0.09	0.06	0.03
7	−0.06	0.04	0.02
8	−0.04	0.03	0.01
9	−0.02	0.02	0.0
10	−0.01	0.01	0.0

*Without second-order terms

Table 3. Second-Order Correction Terms, $g_{i,j}$

i/j	1	2	3	4	5
1	−0.96				
2	−0.4	−0.35			
3	−0.08	−0.1	−0.15		
4	0.0	0.0	−0.05	−0.1	
5	0.0	0.0	0.0	−0.01	−0.06

Remark. Vuthandam et al. (1995) have shown that in the presence of modeling uncertainty, closed-loop performance deteriorates if too short or too long control horizons are used. They have provided a methodology for selecting optimal values for the control horizon length, so that optimal performance bounds are established.

A Case Study

Let us assume that the following equations represent a CSTR, in which the reaction $A \rightarrow B$ takes place.

$$\frac{dC_A(t)}{dt} = \frac{F}{V}(C_{Ai} - C_A(t)) - C_A(t)ke^{(-E/RT(t))} \quad (57)$$

$$\frac{dT(t)}{dt} = \frac{F}{V}(T_i - T(t)) - \frac{\Delta H_R}{c_p\rho}ke^{(-E/RT(t))} - \frac{Q_s(1+u(t))}{c_p\rho V}$$

where $u(t)$ is manipulated input and $T(t)$ is the controlled output; the numerical values of the parameters are given in Table 1. For these values, the CSTR has three steady states. The one we chose is stable.

We want to develop a nonlinear MPC system for the preceding reactor, in order to illustrate the main theory we developed. We will compare two MPC controllers: an ENLDMC and an ELDMC. Both controllers use the same kind of objective function (Eq. 1), and the same constraints, except for the process model. We will show that

- The ENLDMC, designed according to the theory developed in this article, results in a robustly stable closed-loop;
- The robust performance of ENLDMC is superior to that of ELDMC.

The constraints are: $0.35 \geq u(t) \geq -0.35$, and $|du/dt| \leq 0.14$ h^{-1}.

A typical settling time for this CSTR is 10 h, hence, for simplification purposes, we chose $N = 10$.

A linear input/output model {relating u to $y = 100[T(t) - T_s]/T_s$} was obtained from random step and pulse tests, in the least-squares sense (Table 2). Then necessary second-order terms (Tables 3 and 4) were added to improve the accuracy of the model.

As can be seen in Table 2, the addition of the second-order correction terms reduced the uncertainty for the linear part

Table 4. Second-Order Uncertainty, $E_{i,j}$

i/j	1	2	3	4	5
1	0.05				
2	0.02	0.04			
3	0.01	0.01	0.03		
4	0.0	0.0	0.01	0.02	
5	0.0	0.0	0.0	0.0	0.01

Table 5. Performance vs. m, for ELDMC and ENLDMC

m	r_0,ELDMC	P,ELDMC	r_0,ENLDMC	P,ENLDMC
4	6.08	38.713	3.74	35.407
5	7.12	38.705	4.44	35.407
6	8.16	38.711	5.14	35.406
7	9.20	38.713	5.83	36.258
8	10.25	38.713	6.53	36.258
9	11.29	39.400	7.23	36.258
10	12.33	39.418	7.93	36.259

of the model. Uncertainty bounds on both the linear and Volterra model parameters were heuristically developed by trial and error as follows: A number of simulations were run with random inputs to the model Eqs. 57. Then optimal values for the linear and Volterra model parameters were determined. After several simulations an envelope was found for each model parameter. This is in line with the ideas expressed in Chen et al. (1992). We want to emphasize that the identification procedure is not the focus of this work, and more sophisticated methods may certainly be used.

Based on the feasibility condition (Eq. 15), the control horizon length should satisfy the inequality, $m \geq 4$. The move suppression weights $\{r_i\}_{i=0}^{m}$ for the ELDMC and ENLDMC controllers were selected so as to guarantee robust closed-loop stability. For ELDMC the results in Genceli and Nikolaou (1993) were applied, whereas Eqs. 21 and 22 were used for ENLDMC. The move suppression weights and corresponding performance index P for both ELDMC and ENLDMC are shown in Table 5.

Table 5 shows that the optimal control horizon lengths for ELDMC and ENLDMC are $m = 5$ and $m = 6$, with corresponding move suppression weights $r_i = r_0 = 7.12$ and $r_i = r_0 = 5.14$, respectively. A setpoint change from 274.5C to 290C was considered for both the ENLDMC and ELDMC closed loops. Figure 1 shows the prediction error for both controllers. The ENLDMC controller predicts much more accurately than ELDMC does. Figure 2 shows the manipulated input actions. Figure 3 shows the response of the CSTR to the preceding setpoint change. Clearly, the response is improved when ENLDMC rather than ELDMC is used. However, the linearizing effect of feedback should also be noticed. Indeed, although the nonlinear model predicts much better than the linear one, the nonlinear controller performance is only modestly superior to that of the linear controller.

Remark. Prediction with second-order Volterra series is more accurate than prediction with a linear model. This, by itself, improves controller performance. But more importantly, adding second-order terms to the linear model reduces the uncertainty on the first-order terms, which allows the use of smaller move suppression weights that guarantee robust stability. Smaller values for the move suppression weights result in further improvement of the closed-loop robust performance.

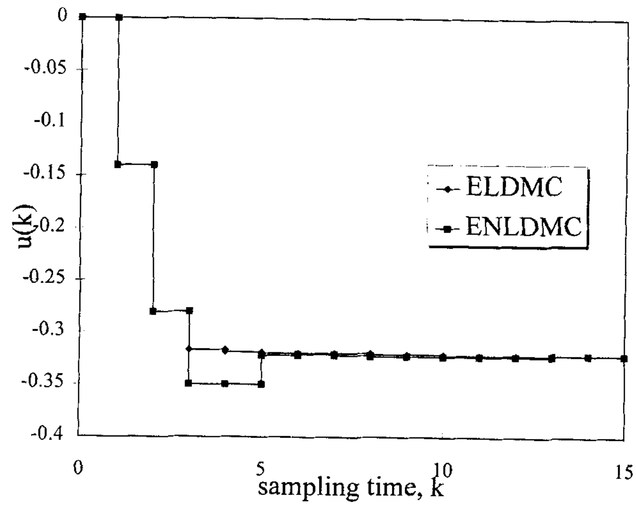

Figure 2. Manipulated input moves.

Discussion and Conclusions

In this work we developed sufficient conditions for robust stability and performance of ENLDMC systems using second-order Volterra series with parametric uncertainty in the time domain. These conditions can be used both for analysis and more importantly, synthesis of nonlinear model-predictive controllers. A case study was presented in order to elucidate these issues. While the results presented here can be used in practice, it remains to be seen how conservative the

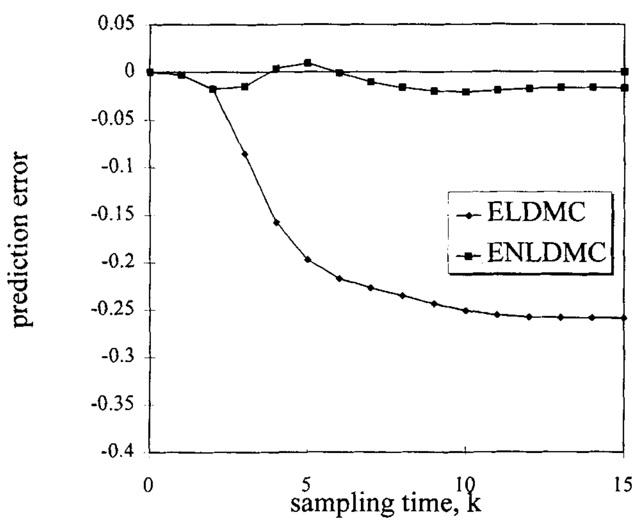

Figure 1. Prediction error.

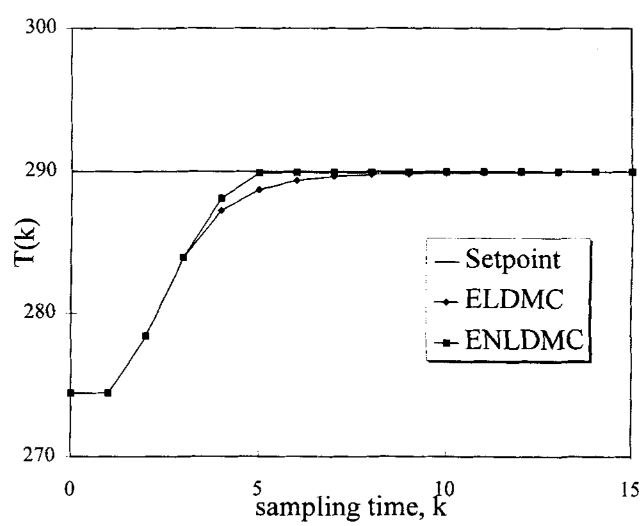

Figure 3. Closed-loop response.

proposed sufficient conditions are. It would be desirable to develop both necessary and sufficient conditions for robust stability and performance. In addition, it would also be desirable to develop a methodology for integrated process modeling and controller design for nonlinear systems, with the objective of getting enough but not redundant modeling accuracy for a certain level of robust closed-loop performance. This is the subject of our current research.

Acknowledgment

Partial financial support from Shell Development (Unrestricted grant) and the Texas Advanced Technology Program-Development (grant No. 999903-321) are gratefully acknowledged.

Notation

d = output additive disturbance
e = model error in first-order terms
$\mathbf{e}$ = model error in second-order terms
E = maximum absolute model error in first-order terms
$\mathbf{E}$ = maximum absolute model error in second-order terms
$f(\cdot)$ = a dummy function bounding the disturbance
g = first-order model coefficient
$\mathbf{g}$ = second-order model coefficient
G = linear part of model gain
$\mathbf{G}$ = nonlinear part of model gain
$J(\cdot)$ = cost functional
N = number of first-order coefficients in the model
K = number of second-order coefficients in the model
M = sampling time that disturbance reaches steady state after
w = soft output constraint tuning parameter

Greek letters

δ = a positive small number
Δ = difference operator: $\Delta x(k) = x(k) - x(k-1)$
ϵ = soft constraint relaxation factor
$\Phi(\cdot)$ = optimal value of a specific cost function

Subscripts and Superscripts

max = maximum limit
min = minimum limit
s = steady state
sp = setpoint

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Appendix: Solving ENLDMC

In this appendix we show how the on-line nonlinear optimization problem

$$\min_{u(k),\ldots,u(k+m),\epsilon(k+1),\ldots,\epsilon(k+p)} \sum_{i=1}^{p} |\bar{y}(k+i) - y^{sp}| + w\epsilon(k+i) + \sum_{i=0}^{p} r_i|\Delta u(k+i)| \quad \text{(A1)}$$

subject to

$$\bar{y}(k+i) = \bar{d} + \sum_{j=1}^{N} g_j u(k+i-j) + \sum_{j_1=1}^{K}\sum_{j_2=1}^{K} g_{j_1,j_2} u(k+i-j_1)u(k+i-j_2) \quad \text{(A2)}$$

$$\bar{d} = y(k) - \sum_{j=1}^{N} g_j u(k-j) - \sum_{j_1=1}^{K}\sum_{j_2=1}^{K} g_{j_1,j_2} u(k-j_1)u(k-j_2) \quad \text{(A3)}$$

$$y_{\min} - \epsilon(k+j) \le \bar{y}(k+j) \le y_{\max} + \epsilon(k+j), \epsilon(k+j) \ge 0 \quad j = 1,\ldots,p$$

$$-\Delta u_{\max} \le \Delta u(k+i) \le \Delta u_{\max} \quad i = 0,\ldots,m$$

$$u_{\min} \le u(k+i) \le u_{\max} \quad i = 0,\ldots,m$$

$$u(k+i) = u_\infty \quad i \ge m$$

can be solved through sequential linear programming (cutting plane method).

Let us assume that a good estimate of the solution $\{u^*(k+i)\}_0^m$ at time k is available (we can always shift the optimal solution at time $k-1$, and use it for an initial feasible estimate at time k). Then each second-order term in Eq. A2 can be linearized (when both $i \ge j_1$ and $i \ge j_2$) through Taylor series expansion around $u^*(k+i-j_1)$, $u^*(k+i-j_2)$ as

$$u(k+i-j_1)u(k+i-j_2) \approx u(k+i-j_1)u^*(k+i-j_2) + u^*(k+i-j_1)u(k+i-j_2) - u^*(k+i-j_1)u^*(k+i-j_2).$$

The preceding formulation (Eq. A1) can be transformed into a linear program, after a simple substitution of variables as follows:

$$\min_{\epsilon(k+1),\ldots,\epsilon(k+p),u(k),u(k+1),\ldots,u(k+m),v_1,\ldots,v_p,\mu_0,\ldots,\mu_m} \sum_{j=1}^{p} v_j + w\epsilon(k+j) + \sum_{i=0}^{m} \mu_i \quad \text{(A4)}$$

subject to

$$\bar{y}(k+i) = \bar{d} + \sum_{j=1}^{N} g_j u(k+i-j) + \sum_{j_1=1}^{K}\sum_{j_2=1}^{K} g_{j_1,j_2} u(k+i-j_1)u(k+i-j_2)$$

$$u(k+i-j_1)u(k+i-j_2) = u(k+i-j_1)u^*(k+i-j_2) + u^*(k+i-j_1)u(k+i-j_2) - u^*(k+i-j_1)u^*(k+i-j_2) \text{ if } i \ge j_1 \text{ and } i \ge j_2$$

$$\bar{d} = y(k) - \sum_{j=1}^{N} g_j u(k-j) - \sum_{j_1=1}^{K}\sum_{j_2=1}^{K} g_{j_1,j_2} u(k-j_1)u(k-j_2)$$

$$\Delta u_{\max} \ge \Delta u(k+i) \ge -\Delta u_{\max}, \quad i = 0,1,2,\ldots,m$$

$$u_{\max} \ge u(k+i) \ge u_{\min}, \quad i = 0,1,2,\ldots,m$$

$$y_{\max} + \epsilon(k+j) \ge \bar{y}(k+j) \ge y_{\min} - \epsilon(k+j), \quad \epsilon(k+j) \ge 0, \quad j = 1,2,\ldots,p$$

$$-v_j \le \bar{y}(k+j) - y^{sp} \le v_j, \quad v_j \ge 0, \quad j = 1,2,\ldots,p$$

$$-\mu_i \le r_i \Delta u(k+i) \le \mu_i, \quad \mu_i \ge 0, \quad i = 0,1,2,\ldots,m$$

$$u(k+m+i) = u_\infty, \quad i \ge 0 \quad \text{(end-condition)}.$$

At any time step k the preceding on-line linear program is solved (iterated) until $\{u^*(k+i)\}_{i=0}^m$ converges.

Manuscript received May 6, 1994, and revision received Nov. 7, 1994.

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